

## Remarks on gauge vortex scattering

R. MacKenzie

*Laboratoire de Physique Nucléaire, Université de Montréal C.P. 6128, succ. centre-ville,  
Montréal, Québec, Canada, H3C 3J7*

### Abstract

In the abelian Higgs model, among other situations, it has recently been realized that the head-on scattering of  $n$  solitons distributed symmetrically around the point of scattering is by an angle  $\pi/n$ , independent of various details of the scattering. In this note, it is first observed that this result is in fact not entirely surprising: the above is one of only two possible outcomes. Then, a generalization of an argument given by Ruback for the case of two gauge theory vortices in the Bogomol'nyi limit is used to show that in the geodesic approximation the above result follows from purely geometric considerations.

hep-th/9503044

The scattering of topological solitons exhibits many interesting and surprising features in a wide variety of situations.<sup>1</sup> For instance, the scattering of monopoles [1], of vortices [2], of skyrmions [3] and of baby skyrmions [4] have many common aspects, such as the fact that, at low energies, head-on scattering in many situations is at ninety degrees. This has been seen both analytically and numerically. Generalizations have been studied wherein the scattering of several solitons has been considered [5–8]. Typically, the initial condition chosen is the most symmetric one possible: one considers any number  $n$  of solitons directed with equal energies towards the origin with relative angular separations  $2\pi/n$ , starting at equal distances, with zero impact parameter (which could be described as an  $n$ -soliton head-on collision). The outcome of such scatterings is rather beautiful: the radii along which the solitons leave after scattering are rotated relative to the incident angles by an angle of  $\pi/n$ . Thus, for instance, if four particles are incident along the positive and negative  $x$  and  $y$  axes, after scattering, the particles will leave along the four axes  $x = \pm y$ , while if three particles are incident with polar angles  $\pi/3$ ,  $\pi$  and  $5\pi/3$ , they scatter along radii of polar angles  $0$ ,  $2\pi/3$  and  $4\pi/3$ .

While at first sight surprising, this result *almost* follows directly from purely symmetric considerations. Let us consider the possible outcomes of such a 2+1-dimensional “experiment” with  $n$  solitons incident at angular separations of  $2\pi/n$ . First, due to the very nature of solitons, the winding number cannot change. If no bound states of solitons exist, there must necessarily be  $n$  distinct solitons after the scattering. Given that the initial configuration is symmetric under rotation by  $2\pi/n$  and under reflections (that is, the symmetry group of a regular  $n$ -gon), the only possible final states which also have this symmetry are one where the solitons leave along the same radii as the incident ones, or one where the final radii are situated midway between the incident ones. It is convenient to adopt a slightly

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<sup>1</sup>We use the “physicist’s definition” of soliton; we are not concerned here with integrability, elasticity of scattering, etc.

*unconventional* definition of scattering angle, namely, as the angle between an incoming radial direction and the nearest outgoing radial direction (rather than measuring scattering relative to the straight-through direction). Then the first possibility corresponds to zero scattering angle while the second corresponds to a scattering of  $\pi/n$ .<sup>2</sup>

In what follows, an argument will be given which shows that in the case of slow-moving gauge theory vortices in the Bobomol’nyi limit [9], scattering is necessarily at an angle of  $\pi/n$ . The argument is a straightforward generalization of the result of Ruback [10], who considered the scattering of two vortices in this limit. Following similar reasoning of Manton [11], Ruback first noted that in this limit there are no static forces between vortices: essentially, the force mediated by the gauge particle is cancelled by that mediated by the Higgs particle. This implies that the motion of slow-moving vortices will be along “troughs” in field space which are the static  $n$ -vortex configurations parameterized by the positions of the zeroes of the Higgs field. This space of configurations is a  $2n$ -dimensional subspace of the whole field space and is the “moduli space” of  $n$  vortices,  $\mathcal{M}_n$  [12]. The motion of vortices at low energy is, in fact, geodesic motion with metric determined by the original field theory action, applied to configurations in  $\mathcal{M}_n$ .

In the case of two vortices,  $\mathcal{M}_2$  clearly separates into two two-dimensional subspaces  $\mathcal{C}$  and  $\mathcal{R}$ , representing center-of-mass and relative motion. It is physically obvious that motion in  $\mathcal{C}$  is trivial due to translational invariance; mathematically, this invariance implies that the induced metric on  $\mathcal{M}_2$  is in fact block-diagonal, there being no mixed term between  $\mathcal{C}$  and  $\mathcal{R}$ . Thus, one can deduce the dynamics of two vortices by studying the induced metric

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<sup>2</sup>The reason for adopting this unconventional definition is that with the traditional definition the cases of even or odd number of solitons must be treated separately: the first possibility corresponds to traditional scattering angle 0 or  $\pi/n$  for  $n$  even or odd, respectively, while the second corresponds to traditional scattering angle  $\pi/n$  or 0 for  $n$  even or odd, respectively. Note that, since the solitons are identical, scattering is in fact only defined modulo  $2\pi/n$ .

in  $\mathcal{R}$  only.

In the case of  $n$  vortices,  $\mathcal{M}_n$  is somewhat more complicated. Two of the  $2n$  dimensions clearly represent center-of-mass motion in a submanifold  $\mathcal{C}$ , leaving a  $2(n-1)$ -dimensional submanifold  $\mathcal{R}$  describing nontrivial relative motion. We can further divide  $\mathcal{R}$  into a 2-dimensional piece  $\mathcal{S}$  which describes “symmetric” configurations where the vortices are equidistant from the origin and separated from one another by relative angles of  $2\pi/n$ , and a  $2(n-2)$ -dimensional piece describing deviations from such symmetric configurations. The choice of coordinates for  $\mathcal{S}$  can be taken to be  $(\rho, \psi)$ , the distance from the origin and the angular position of a vortex relative to some reference direction (the  $x$ -axis, say). The choice for  $\mathcal{R} \setminus \mathcal{S}$  is much less obvious, but again symmetry comes to the rescue [6]: since configurations in  $\mathcal{S}$  are symmetric under discrete rotations, any movement starting in  $\mathcal{S}$  with initial velocity along  $\mathcal{S}$  will remain in  $\mathcal{S}$ , since any “orthogonal” motion would violate the rotational symmetry. This implies that a choice of coordinates for  $\mathcal{R}$  exists for which the induced metric is block diagonal with no mixing between  $\mathcal{S}$  and  $\mathcal{R} \setminus \mathcal{S}$ . With this choice, we are free to consider just the motion within  $\mathcal{S}$ .

For the induced metric in  $\mathcal{S}$ , with coordinates  $(\rho, \psi)$  described above, rotational symmetry dictates that the metric is

$$ds^2 = f(\rho)d\rho^2 + g(\rho)d\psi^2. \quad (1)$$

Following Ruback, static configurations differing by  $\psi \rightarrow \psi + 2\pi/n$  are identical, so it is tempting to impose this periodicity on the angular variable in  $\mathcal{S}$ . One must be cautious, however, since *a priori* one could arrive at a conic singularity at the origin, which would render geodesic motion ambiguous at that point. A straightforward generalization of Ruback’s calculation for the case of two vortices shows that near the origin the metric for the submanifold  $\mathcal{S}$  is given by

$$ds^2 \propto d\rho^2 + n^2\rho^2 d\psi^2, \quad (2)$$

so that encircling the origin yields the flat-space relation between circumference and radius: there is no conic singularity if  $\psi$  and  $\psi + 2\pi/n$  are identified.

Now consider geodesic motion in  $\mathcal{S}$  which passes through the origin. One has initially  $\psi = 0$ , say, and  $\rho$  decreasing. This represents an  $n$ -vortex head-on collision, as described above. As  $\rho$  decreases to zero, one passes to the other side of the origin in  $\mathcal{S}$  along a straight line in  $\mathcal{S}$ , and emerges with  $\rho$  increasing and  $\psi$  changed by half its range, *i.e.*,  $\psi = \pi/n$ . In configuration space, this is exactly the type of scattering by angle  $\pi/n$  as seen in previous work.

This behaviour is seen outside the Bogomol'nyi limit and beyond the geodesic approximation for vortices (until vortex-antivortex creation becomes possible) [7]. In contrast, it is interesting to note that for the case of baby skyrmions, *fast* skyrmions scatter at  $\pi/n$ , while in the geodesic approximation the scattering is by angle zero: there is a repulsive barrier to overcome before nontrivial scattering takes place [8].

In summary, the motion of gauge theory vortices in the Bogomol'nyi limit and in the geodesic approximation clearly shows that for  $n$ -vortex head-on collisions, the scattering is necessarily by an angle  $\pi/n$ , as has been seen in recent numerical and analytical studies.

I thank T. Gisiger and M. Paranjape for useful discussions, and Cairo for inspiration. This work was supported by the Natural Science and Engineering Research Council of Canada.

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